MATH 2260 Calculus II
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Name: Answers
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Practice Quiz No. 3

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 Evaluate the integral

$$\int \cos^2(x) \sin^3(x) dx$$

$$\cos^2(x) + \sin^2(x) = 1 \implies \sin^2(x) = 1 - \cos^2(x)$$

$$\implies \int \cos^2(x)(1-\cos^2(x)) \sin(x) dx$$

$$u = \cos(x), \quad du = -\sin(x) dx$$

$$= -\int u^2 - u^4 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C$$

Problem 2 Evaluate the integral

$$\frac{-x^2}{2} - 2\ln|4-x^2| + \frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x-2| = \int \frac{x^3+2}{4-x^2} dx$$

$$\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$\implies 1 = A(x-2) + B(x+2) \implies$$

$$A+B = 0, \quad 1 = -2A + 2B, \quad B = \frac{1}{4}, \quad A = -1/4$$

$$\begin{array}{r} -x \\ -x^2+4 \enclose{longdiv}{x^3+2} \\ -(+x^3-4x) \\ \hline 4x+2 \end{array}$$

$$\implies \frac{x^3+2}{4-x^2} = -x + \frac{4x+2}{4-x^2}$$

$$\implies \int \frac{x^3+2}{4-x^2} dx = \int -x + \frac{4x+2}{4-x^2} dx = \frac{-x^2}{2} + \int \frac{4x+2}{4-x^2} dx$$

$$= \frac{-x^2}{2} + \int \frac{4x}{4-x^2} dx + \int \frac{2}{4-x^2} dx$$

$$= -2\int \frac{1}{x^2-4} dx = -2\int \frac{1}{(x+2)(x-2)} dx$$

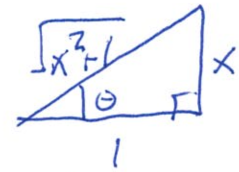
$$= -2\left(\int \frac{-1/4}{x+2} dx + \int \frac{1/4}{(x-2)} dx\right)$$

$$u = 4-x^2, \quad du = -2x\,dx$$

$$= \frac{-x^2}{2} - 2\int \frac{1}{u} du + \frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x-2| \qquad = \frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x-2|$$

Problem 3 Evaluate the integral

$$\int \frac{4}{x^2\sqrt{x^2+1}}dx$$



$x = \tan(\theta)$
$dx = \sec^2(\theta)\,d\theta$

$$\int \frac{4}{x^2\sqrt{x^2+1}}dx = 4\int \frac{1}{\tan^2(\theta)\sqrt{\tan^2(\theta)+1}}\left(\sec^2(\theta)\right)d\theta = 4\int \frac{\sec^2(\theta)}{\tan^2(\theta)\,|\sec(\theta)|}\,d\theta$$

$$= 4\int \frac{\sec(\theta)}{\tan^2(\theta)}d\theta = 4\int \frac{1/\cos(\theta)}{\sin^2(\theta)/\cos^2(\theta)}d\theta = 4\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta$$

Assuming $\sec(\theta) > 0$

$u = \sin(\theta) \quad du = \cos(\theta)\,d\theta$

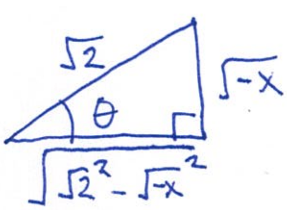
$$= 4\int \frac{1}{u^2}du = 4\left(\frac{-1}{\sin(\theta)}\right) + C = \frac{-4}{x/\sqrt{x^2+1}} + C$$

Problem 4 Evaluate the integral

$$\int \frac{1}{\sqrt{-2x-x^2}}dx$$

$-2x - x^2 = -x(2+x) \geq 0$ if $-2 \leq x \leq 0$, so $-x$ is $\geq$

$$\Rightarrow \int \frac{1}{\sqrt{-2x-x^2}}dx = \int \frac{1}{\sqrt{-x}\sqrt{2+x}}dx = \int \frac{1}{\sqrt{-x}\sqrt{(\sqrt{2})^2-(\sqrt{-x})^2}}dx$$



$$\frac{\sqrt{-x}}{\sqrt{2}} = \sin(\theta) \Rightarrow \frac{-x}{2} = \sin^2(\theta) \Rightarrow x = -2\sin^2(\theta)$$

$$\Rightarrow dx = -4\sin(\theta)\cos(\theta)\,d\theta \text{ and } \sqrt{-x} = \sqrt{2}\sin(\theta)$$

$$\Rightarrow \int \frac{1}{\sqrt{-x}\sqrt{2+x}}dx = \int \frac{1}{\sqrt{-x}\sqrt{\sqrt{2}^2-\sqrt{-x}^2}}dx = \int \frac{-4\sin(\theta)\cos(\theta)}{(\sqrt{2}\sin(\theta))(2-2\sin^2(\theta))^{1/2}}d\theta$$

$$= \frac{-4}{2}\int \frac{\cos(\theta)}{|\cos(\theta)|}d\theta = -2\int d\theta = -2\theta + C = -2\sin^{-1}\left(\frac{\sqrt{-x}}{2}\right) + C$$

Assuming $\cos(\theta) > 0$

Problem 5 Solve the differential equation

$$\frac{dx}{dt} = x(1-x)$$

$$\Rightarrow \int \frac{1}{x(1-x)}\,dx = \int dt = t + C \quad , \quad \frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$$\Rightarrow 1 = A(1-x) + B(x) \Rightarrow A = 1,\ B = 1$$

$$\Rightarrow \int \frac{1}{x(1-x)}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{1-x}\,dx = \ln(|x|) - \ln(|1-x|) = t + c$$

$$\Rightarrow \ln\left(\frac{|x|}{|1-x|}\right) = t + C \Rightarrow \left|\frac{x}{1-x}\right| = e^{t+C} \quad , \text{ just assume } 0 < x < 1$$

$$\Rightarrow \frac{x}{1-x} = e^{t+C} \Rightarrow x = (1-x)e^{t+c} = e^{t+c} - xe^{t+c}$$

$$\Rightarrow x + xe^{t+c} = e^{t+c} \Rightarrow x(1+e^{t+c}) = e^{t+c} \Rightarrow$$

$$x = \frac{e^{t+c}}{1+e^{t+c}} = \frac{1}{1+e^{-t-}}$$

Logistic function

Problem 6 Evaluate the integral

$$\int \frac{1}{x^3 + x^2 - 12x}\,dx$$

$$x^3 + x^2 - 12x = x(x^2 + x - 12) = x(x+4)(x-3)$$

$$\Rightarrow \frac{1}{x(x+4)(x-3)} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-3}$$

$$\Rightarrow 1 = A(x+4)(x-3) + B(x)(x-3) + C(x)(x+4)$$

$$x = 0 \Rightarrow 1 = A(4)(-3) \Rightarrow A = \frac{-1}{12}$$

$$x = -4 \Rightarrow 1 = B(-4)(-7) \Rightarrow B = \frac{1}{28}$$

$$x = 3 \Rightarrow 1 = C(3)(7) \Rightarrow C = \frac{1}{21}$$

$$\Rightarrow \int \frac{1}{x^3+x^2-12x}\,dx = \int \frac{-1/12}{x}\,dx + \int \frac{1/28}{x+4}\,dx + \int \frac{1/21}{x-3}\,dx$$

$$= \frac{-1}{12}\ln(|x|) + \frac{1}{28}\ln(|x+4|) + \frac{1}{21}\ln(|x-3|) + C$$

Problem 7 Evaluate the integral

$$\int \frac{1}{36-x^2}dx$$

$$\frac{1}{36-x^2} = \frac{1}{(6-x)(6+x)} = \frac{A}{6-x} + \frac{B}{6+x}$$

$$\Rightarrow \quad 1 = A(6+x) + B(6-x)$$

$$x=6 \Rightarrow 1 = 12A \to A = \frac{1}{12}$$

$$x=-6 \Rightarrow 1 = 12B \Rightarrow B = \frac{1}{12}$$

$$u = 6-x$$
$$du = -dx$$

$$\Rightarrow \int \frac{1}{36-x^2}dx = \int \frac{1/12}{6-x} + \frac{1/12}{6+x}dx = \frac{1}{12}\int \frac{-1}{u}du + \frac{1}{12}\ln|6+x|$$

$$= \frac{1}{12}\left(-\ln|6-x| + \ln|6+x|\right) + C = \frac{1}{12}\ln\left|\frac{6+x}{6-x}\right| + C$$

Problem 8 Evaluate the integral

$$\int \frac{\sin^2(t)}{\cos^2(t)}dt$$

$$\cos^2(t) + \sin^2(t) = 1$$
$$\sin^2(t) = 1 - \cos^2(t)$$

$$\Rightarrow \int \frac{\sin^2(t)}{\cos^2(t)}dt = \int \frac{1-\cos^2(t)}{\cos^2(t)}dt = \int \sec^2(t)\,dt - \int 1\,dt$$

$$= \tan(t) - t + C$$

Problem 9 Evaluate the integral

$$\int t\cos(2t+1)dt$$

$$\int \underbrace{t}_{g}\, \underbrace{\cos(2t+1)}_{f'}\, dt = \frac{1}{2}\sin(2t+1)t - \int \frac{1}{2}\sin(2t+1)\,dt$$

$$= \frac{1}{2}\sin(2t+1)t + \frac{1}{4}\cos(2t+1) + C$$

$$f = \int \cos(2t+1)\,dt$$
$$= \frac{1}{2}\sin(2t+1)$$

Problem 10 Evaluate the integral

$$\int_{\pi/4}^{\pi/2} \sqrt{1+\cos(4x)}dx$$

Remember: $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

so $\cos^2(2x) = \dfrac{1+\cos(4x)}{2} \Rightarrow 1+\cos(4x) = 2\cos^2(2x$

$$\Rightarrow \int_{\pi/4}^{\pi/2} \sqrt{1+\cos(4x)}\,dx = \int_{\pi/4}^{\pi/2} \sqrt{2\cos^2(2x)}\,dx = \sqrt{2}\int_{\pi/4}^{\pi/2} |\cos(2x)|\,dx$$

On $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $2x$ goes from $\frac{\pi}{2}$ to π and $\cos(x)$ is <u>negative</u> on this range, so $|\cos(2x)| = -\cos($

$$\Rightarrow = \sqrt{2}\int_{\pi/4}^{\pi/2} -\cos(2x)\,dx = -\frac{\sqrt{2}}{2}\Big[\sin(2x)\Big]_{\pi/4}^{\pi/2} = -\frac{1}{\sqrt{2}}\left(\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right) =$$

Problem 11 Evaluate the integral

$$\int x^3 e^{x^2}\, dx$$

$$= \frac{1}{2}\int \underbrace{x^2}_{g}\underbrace{\left(2x e^{x^2}\right)}_{f'} dx = \frac{1}{2}\left(x^2 e^{x^2} - \int 2x e^{x^2} dx\right)$$

$$= \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C$$

$$f' = 2x e^{x^2}$$

$$\Rightarrow f = \int 2x e^{x^2} dx$$

$$= e^{x^2}$$

Problem 12 Evaluate the integral

$$\int \frac{\arctan(x)}{x^2} dx$$

$$\int \frac{\tan^{-1}(x)}{x^2} dx = \int \underbrace{\frac{1}{x^2}}_{f'} \underbrace{\tan^{-1}(x)}_{g} dx = \frac{-\tan^{-1}(x)}{x} - \int \left(\frac{-1}{x}\right)\left(\frac{1}{1+x^2}\right) dx$$

$$f = \frac{-1}{x}, \quad g' = \frac{1}{1+x^2}$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$\Rightarrow 1 = A(1+x^2) + (Bx+C)(x)$$

$$1 = (A+B)x^2 + Cx + A$$

$$\Rightarrow A=1, \quad B=-1, \quad C=0$$

$$= \frac{-\tan^{-1}(x)}{x} + \int \frac{1}{x(1+x^2)} dx$$

$$= \frac{-\tan^{-1}(x)}{x} + \int \frac{1}{x} dx + \int \frac{-x}{1+x^2} dx$$

$$= \frac{-\tan^{-1}(x)}{x} + \ln(|x|) - \frac{1}{2}\int \frac{1}{1+x^2} 2x\, dx$$

$$u = 1+x^2, \quad du = 2x\, dx$$

$$= \frac{-\tan^{-1}(x)}{x} + \ln(|x|) - \frac{1}{2}\ln(|1+x^2|) + c$$